Resilient Containment Under Time-Varying Networks With Relaxed Graph Robustness

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***Abstract*—This paper investigates the resilient containment control problem for leader-follower MASs in time-invariant and time-varying digraphs. Despite the existence of some noncooperative agents in the network, the cooperative followers are expected to converge to the safety interval constructed by the cooperative leaders. Specifically, to defend against malicious attacks and achieve the objective of resilient containment, each cooperative follower disregards the most suspicious values in its in-neighbor set and utilizes the retained values for state update. However, resilient containment is usually achieved at the cost of stringent graph conditions. In our work, with the introduction of a novel graph-theoretic property, namely the *strongly trusted robustness*, a small subset of agents is set as trusted nodes and the graph robustness requirement for resilient containment is relaxed. The constraint on the minimum number of leaders is also relaxed through this operation. Moreover, this novel property is extended to time-varying networks, for which the notion of *jointly and strongly trusted robustness* is proposed. This notion further relaxes the requirement that the digraph should satisfy certain graph conditions at each time step, thus reducing the communication burden. Numerical simulations are provided to validate the theoretical results.**

***Index Terms*—Resilient containment, trusted nodes, network relaxation, time-varying graph.**

I. Introduction

The Distributed cooperative control of multi-agent systems (MASs) has gained great research interest in recent years [1], [2], [3], [4], [5]. The ultimate goal of MASs is to achieve a global objective through the interaction of multiple agents, for which the single agent cannot achieve individually [6], [7], [8]. Containment control, as a method of the multi-agent cooperative control, is an important area of research in the field of networked systems and has attracted remarkable attention in the past decade [9], [10], [11], [12], [13], [14]. Different from achieving consensus, containment control generally requires a leader-follower structure and seeks for appropriate distributed algorithms to drive the follower agents (followers) to move within the safety interval constructed by the leader agents (leaders). However, lack of global situational awareness makes distributed MASs vulnerable to malicious attacks or faults, that is, agents that may have suffered a cyber attack or may have encountered a fault would lose the capability of executing the prescribed control protocols [15]. Such attacks can undermine the cooperative agents in the network, destroy the achievement of state containment among the cooperative individuals and even lead to the overall system paralysis [16], [17], [18]. Thus, it is critical to study resilient control for MASs by designing reliable and secure algorithms to achieve the desired system containment in presence of malicious attacks.

Generally, there are two categories of methods for addressing resilient control problems. One category is based on the idea of fault identification and isolation, i.e., noncooperative agents in the network are firstly identified, and then isolated [19], [20], [21]. For distributed MASs, however, identifying and isolating noncooperative agents is challenging since these processes require agents to handle massive information. It is also impractical to identify all malicious attacks in large-scale distributed MASs. The second category is a family of algorithms called Mean-Subsequence-Reduced (MSR), which has been proposed to initially achieve resilient consensus [22], [23], [24], [25], [26], with each cooperative agent ignoring agents presenting the most significant deviation. These discrete-time algorithms include the weighted MSR (W-MSR) [22], double-integrator position-based MSR (DP-MSR) [23], event-based MSR (E-MSR) [24], dynamic event-triggered MSR (DE-MSR) [25], and trusted-region subsequence reduction (TSR) [26] algorithms. However, these promising results merely guarantee that the cooperative agents converge to a constrained safety interval, which can only be determined by the initial state values of all cooperative agents in the network. In this paper, we are going to relax this constraint via a leader-follower structure.

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For containment control, the idea of MSR has also been shown to be an essential approach to ensure resilience [27], [28], [29]. In [27], the authors studied the resilient containment problem for MASs and adopted the idea of MSR to ensure that all cooperative followers move within the safety interval constructed by cooperative leaders. Subsequently, the problem is extended to multi-robot systems (MRSs) and the dynamic resilient containment control problem for continuous-time MRSs is investigated in [28], where a distributed sign-based local interaction protocol was proposed to drive a set of robots toward the target region and keep them within the dynamic containment area regardless of the persistent influence of anonymous adversarial robots. In [29], the multi-dimensional resilient containment problem was considered and a novel distributed local interaction protocol was proposed, which was secure against the influence of a finite set of anonymous adversarial agents. However, the aforementioned methods have high requirements for network topology with stringent *graph robustness*. With the increasing number of noncooperative agents, the network will become extremely complex.

The notion of graph robustness was defined in [22] together with the resilient control and was extended to *strong graph robustness* under the leader-follower structure [30]. Current results on resilient containment built on stringent conditions on graph robustness [27], [29]. Specifically, under the f-local attack model (each agent can be influenced at maximum by f noncooperative agents in its in-neighbors), resilient containment can be achieved if the topology of the network is strongly $(3f+1)$-robust [27]. For resilient containment in n-dimensional space, the0 network should be strongly $((n+1)f+1)$-robust [29]. Furthermore, it was revealed in [31] that a strongly robust network induces a lower bound on the number of leaders, e.g., a strongly $(3f+1)$-robust network should possess at least $3f+1$ leader agents to overcome the influence of malicious attacks. These facts indicate that it is urgent to relax the stringent graph requirement for achieving resilient containment with the help of some external conditions.

Another method to model robust networks is to incorporate trusted nodes, which were defined in [32]. This kind of agent is assumed sufficiently secure such that they cannot be compromised by malicious attacks. In [32], the notion of trusted nodes was incorporated to ensure resilient consensus for MASs regardless of any number of adversarial attacks. The idea was then extended to solve distributed optimization problems [33]. The paper [34] applied trusted nodes to improve the connectivity and robustness of the network. In [35], trusted nodes were introduced to achieve resilient consensus for MASs in switching networks. Despite these promising results, most of them require all agents in the network to be able to identify trusted nodes, while the set of trusted nodes needs to form a connected dominating set.

Compared to time-invariant networks, time-varying networks are more common which relax the requirement that the digraph should satisfy certain graph conditions at each time step. Currently, there are only a few studies that investigated resilient control under time-varying networks [35], [36], [37], [38]. In [36], the time window T was introduced and a sliding window Weighted-MSR (SW-MSR) algorithm was developed to guarantee resilience for networks of agents with time-varying graphs. The authors of [37] extended the results in [36] and achieved resilient consensus for second-order MASs in time-varying communication networks. In [38], the notion of *joint robustness* was introduced and the necessary and sufficient conditions were provided to ensure resilient consensus for first-order and second-order MASs in switching digraphs. With the comprehensive consideration of trusted nodes and time windows, the paper [35] achieved resilient consensus for heterogeneous MASs in time-varying digraphs. Nevertheless, existing studies mainly focus on leaderless networks. For leader-follower networks, the results are not applicable since strongly robust networks are required.

Motivated by the above observations, this study aims to design a distributed controller for all cooperative agents to achieve resilient containment, such that each cooperative follower moves within the safety interval constructed by leaders in presence of malicious attacks. To the best of our knowledge, previous methods are established on networks with high robustness requirements. In this work, we prove that by setting leaders as trusted nodes, the graph robustness for resilient containment is relaxed, and the constraint on the minimum number of leaders is also reduced. Furthermore, no identification for trusted nodes or additional graph condition is required. Finally, resilient containment under time-varying networks is achieved through a novel graph condition.

The main contributions of this study are highlighted as follows.

1) We study the containment control problem for MASs in presence of malicious attacks. The MSR-based idea is adopted to eliminate the most suspicious values in the in-neighbor set of cooperative followers and drive them into the preset safety interval, thus enhancing the system security. In contrast to the studies [22], [23], [24], [25], [26] that merely pursue a fixed safety interval, which is determined by the initial state values of all cooperative agents in the network, the safety interval mentioned in this study is determined by the state values of leaders and can be adjusted by leaders flexibly.
2) Compared with the existing resilient algorithms without trusted nodes [22], [27], [29] or with trusted nodes [32], [34], the proposed notion of *strongly trusted robustness* achieves a good balance between the cost of setting trusted nodes and the cost of constructing robust graphs. With this novel graph notion, the sufficient condition for ensuring resilient containment is relaxed from strongly $(3f+1)$-robustness to strongly trusted $(2f+1)$-robustness, which means that fewer directed edges are required. Furthermore, the constraint on the minimum number of leaders is also relaxed from $3f+1$ to $2f+1$. Both of these two relaxations are beneficial for multi-agent networks.
3) Different from the existing resilient algorithms in time-invariant systems [22], [30], [39], the resilient containment problem under time-varying networks is studied in this paper. The proposed notion of *jointly and strongly trusted*

robustness enables agents in the network to sparsely connect with other agents at each time step, thus reducing the communication burden. In contrast to the current research on time-varying networks [36], [40] that necessitates each cooperative follower to store the in-neighbor information within T time steps, prior information for the time window T is not required in our work. This fact means that cooperative followers do not need to store a massive amount of information. The data-storage burden for each cooperative follower is thereby mitigated.

The rest of the paper is organized as follows. Section II introduces some preliminaries on graph theory, together with formulating the resilient containment problem. Section III presents the main results for achieving resilient containment in time-invariant and time-varying networks. We validate the main results through numerical examples in Section V. Finally, Section VI concludes this paper and presents future research directions.

II. Preliminaries and Problem Formulation

A. Preliminaries on Graphs

Consider a time-varying digraph $\mathcal{G}[t] = (\mathcal{V}, \mathcal{E}[t])$ with a leader-follower structure. The set of nodes[1] is given by $\mathcal{V}$. The set of time-varying edges is denoted by $\mathcal{E}[t] \subseteq \mathcal{V} \times \mathcal{V}$. The edge $(j,i) \in \mathcal{E}[t]$ indicates that node i has access to information from node j. Since $\mathcal{G}[t]$ is a digraph, we define the sets of in-neighbors and out-neighbors as $\mathcal{V}_i^+[t] = \{j \in \mathcal{V} | (j,i) \in \mathcal{E}[t]\}$ and $\mathcal{V}_i^-[t] = \{j \in \mathcal{V} | (i,j) \in \mathcal{E}[t]\}$, respectively. Furthermore, if $\mathcal{E}[t_i] = \mathcal{E}[t_j]\ \forall t_i, t_j \in \mathbb{Z}_{\geq 0}$, then $\mathcal{G}[t]$ becomes the time-invariant digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

B. Leader-Follower MAS

In our work, the leader-follower MAS contains two types of agents: 1) leader agents; and 2) follower agents, with the former propagating a safety interval and the latter moving within the interval to achieve containment. The sets of leader and follower agents are defined as $\mathcal{L}$ and $\mathcal{F}$, respectively. Denote $x_i[t] \in \mathbb{R}$ as the state value of agent i and $x_j^i[t] \in \mathbb{R}$ as the value transmitted from agent j to agent i at time step t, respectively. The state updates for cooperative leaders and followers are based on prescribed rules, which are presented as

$$\begin{aligned} x_i[t+1] &= f_i\left(\{x_j^i[t]\}\right), \\ x_l[t+1] &= c_l, \end{aligned} \tag{1}$$

where $i \in \mathcal{F}, j \in \mathcal{V}_i^+[t], l \in \mathcal{L}, t \in \mathbb{Z}_{\geq 0}$. Note that $f_i(\cdot)$ can be an arbitrary function, while c_l denotes the state value of Leader l, which remains constant at certain time intervals but may suddenly change at some time steps. In addition, there may exist some noncooperative agents in the network that deviate from the normal update rule (1). Instead, they apply some other rules $f_i'(\cdot)$ to update and disseminate malicious information to all of their out-neighbors. Such misbehaviors may diminish the effectiveness of control strategies and disrupt system security. The precise definitions of *cooperative* and *noncooperative* agents will be given in the next subsection.

[1] In this paper, the terms of *node* and *agent* are adopted exchangeably.

C. Attack Model

This paper considers the containment control in an adversarial environment, where some of the agents are noncooperative. Such misbehaving ones may deteriorate the system performance by misleading the cooperative followers to leave the convex hull of cooperative leaders, or destroying the convex formation of leaders. In addition, a small subset of cooperative agents are allowed to be equipped with high security, which means that attacking these nodes is sufficiently difficult. We can safely assume that this small group of agents are trusted and cannot be compromised. The following definitions describe the behaviors of different agents in the MAS.

Definition 1 ([22]): An agent is said to be cooperative if it utilizes the prescribed rule to update its state. We denote the set of cooperative agents as $\mathcal{C} \subseteq \mathcal{V}$.

Definition 2 ([22]): An agent is said to be noncooperative if it does not follow the prescribed update rule but utilizes some other rule $f_i'(\cdot)$ for update at some time steps. We denote the set of noncooperative agents as $\mathcal{A} := \mathcal{V} \backslash \mathcal{C}$.

Definition 3 ([32]): An agent is said to be trusted if it follows the prescribed rule and cannot be compromised by malicious attacks. We denote the set of trusted agents as $\mathcal{T} \subseteq \mathcal{C}$.

By synthesizing Definition 1 with the leader-follower network structure, we further denote the sets of cooperative leaders and followers as $\mathcal{L}_c$ and $\mathcal{F}_c$, respectively.

In our setting, the attacker (noncooperative agent) has the knowledge about the entire topology of the network and the distributed controller deployed on each agent. In addition, we consider a limitation on the number of noncooperative agents in the in-neighbor set of each cooperative agent as below.

Definition 4 ([22]): A multi-agent network satisfies the f-local attack model if there exist at most f noncooperative nodes in the in-neighbor set of each cooperative agent, i.e., $|\mathcal{V}_i^+[t] \cap \mathcal{A}| \leq f,\ \forall i \in \mathcal{V} \backslash \mathcal{A},\ \forall t \in \mathbb{Z}_{\geq 0}$, where $f \in \mathbb{Z}_{\geq 0}$.

D. Problem Formulation

The resilient containment of the MAS aims to solve the following problem.

Problem 1: Consider a MAS described by a time-varying digraph $\mathcal{G}[t] = (\mathcal{V}, \mathcal{E}[t])$. Suppose that at most f noncooperative agents exist in the in-neighbor set of each agent in the network. Determine graph conditions and design controllers such that

$$\begin{aligned} \lim_{t\to\infty} \max_i x_i[t] - \lim_{t\to\infty} \max_l x_l[t] \leq 0, \\ \lim_{t\to\infty} \min_i x_i[t] - \lim_{t\to\infty} \min_l x_l[t] \geq 0 \end{aligned} \tag{2}$$

for all cooperative followers i and cooperative leaders l.

Remark 1: The objective of Problem 1 can be rewritten as $\lim_{t\to\infty} x_i[t] \in \mathcal{I},\ \forall i \in \mathcal{F}_c$, where $\mathcal{I} = [m_L, M_L]$, $M_L = \max_{l\in\mathcal{L}_c} c_l$ and $m_L = \min_{l\in\mathcal{L}_c} c_l$. In this paper, we assume that all leaders are static and their objective is to propagate reference values and construct the safety interval $\mathcal{I}$. The objective of all cooperative followers is to converge to $\mathcal{I}$ despite the influence of some noncooperative agents in the network. Note that the safety interval $\mathcal{I}$ is completely determined by cooperative leaders and can be set at an arbitrary position. This fact reflects the importance of leader trustiness and promotes Definition 8 of *strongly trusted robustness*, as shown in Section III. In addition, the achievement of resilient containment requires $|\mathcal{L}| \geq 2$, so that the safety interval can be constructed.

E. Review of Resilient Control Concepts

In this subsection, some essential notions with respect to (w.r.t.) sets and graphs are presented to characterize the resilience for the leader-follower MASs.

Definition 5 ([22]): Consider a time-invariant digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A nonempty subset $\mathcal{S} \subseteq \mathcal{V}$ is r-reachable if there exists an agent $i \in \mathcal{S}$ such that $|\mathcal{V}_i^+[t]\mathcal{S}| \geq r$, where $r \in \mathbb{Z}_{>0}$.

Definition 6 ([30]): Consider a time-invariant digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a nonempty subset $\mathcal{S}_1 \subseteq \mathcal{V}$. $\mathcal{G}$ is strongly r-robust w.r.t. $\mathcal{S}_1$ if for any nonempty subset $\mathcal{S}_2 \subseteq \mathcal{V} \backslash \mathcal{S}_1$, $\mathcal{S}_2$ is r-reachable, where $r \in \mathbb{Z}_{>0}$.

In this paper, we consider resilient containment control in both time-invariant and time-varying digraphs. A common time-varying version of Definition 6 is extended as follows.

Definition 7 ([40]): Consider a time-varying digraph $\mathcal{G}[t] = (\mathcal{V}, \mathcal{E}[t])$. $\mathcal{G}[t]$ is strongly (T, t_0, r)-robust w.r.t. a nonempty set $\mathcal{S}_1 \subseteq \mathcal{V}$ if $\mathcal{G}^T[t]$ is strongly r-robust w.r.t. $\mathcal{S}_1$, where $\mathcal{G}^T[t] = \cup_{\tau=0}^{T} \mathcal{G}[t-\tau]$, $T, r, t \in \mathbb{Z}_{>0}$, $t_0 \in \mathbb{Z}$ and $t \geq t_0 + T$.

Definition 7 was proposed and adopted in [40] to achieve resilient consensus for leader-follower MASs in switching digraphs. Nevertheless, this method requires that the agents to store the in-neighbor information within T time steps. This constraint will be eliminated through a novel graph condition in Section IV.

F. Leader-Follower W-MSR Algorithm

We focus on the resilient containment problem for first-order MASs. The discrete-time update scheme is presented as

$$x_i[t+1] = x_i[t] + \sum_{j \in \mathcal{V}_i^+[t]} a_{ij}[t] \left(x_j[t] - x_i[t] \right) \tag{3}$$

where $a_{ij}[t]$ is the weight of edge (j, i) which satisfies $\sum_{j=1}^{|\mathcal{V}|} a_{ij}[t] = 1$. Furthermore, we assume $a_{ij}[t] \in [\omega, 1)$ if $(j, i) \in \mathcal{E}$ and otherwise $a_{ij}[t] = 0$, where $0 < \omega < 1$ refers to a fixed lower bound. The cooperative agents and trusted agents update their state values through the leader-follower weighted Mean-Subsequence-Reduced (W-MSR) algorithm, as outlined in Algorithm 1. Different from the update scheme (3), the cooperative agents and trusted agents will only utilize the state values of in-neighbors retained after Algorithm 1 for update, as expressed in (4).

It is expected that through Algorithm 1, the MAS could be resilient to malicious attacks and achieves containment successfully. In the next section, we will investigate the stringent graph conditions under which cooperative followers resiliently converge to $\mathcal{I}$, despite the misbehavior of noncooperative agents.

Algorithm 1: Leader-Follower W-MSR Algorithm [22].

Require: Upper bound of the number of noncooperative agents f
Initialize the state $x_l[0]$ of cooperative leader l;
Construct $\mathcal{I} = [\min_{l \in \mathcal{L}_c} x_l[0], \max_{l \in \mathcal{L}_c} x_l[0]]$;
Initialize the state $x_i[0]$ of cooperative follower i;
for $t = 0, 1, \ldots$ **do**
  Send $x_i[t]$ to the out-neighbor set $\mathcal{V}_i^-[t]$ of agent i;
  Receive $x_j^i[t]$ from the in-neighbor set $\mathcal{V}_i^+[t]$;
  if less than f of the values in $\mathcal{V}_i^+[t]$ are strictly larger [resp. smaller] than $x_i[t]$ **then**
    Eliminate all the values in $\mathcal{V}_i^+[t]$ that are strictly larger [resp. smaller] than $x_i[t]$;
  else
    Eliminate the f largest [resp. smallest] values in $\mathcal{V}_i^+[t]$;
  end if
  Denote $\mathcal{R}_i^+[t]$ as the set of retained in-neighbors for agent i;
  Update the state of cooperative follower i as

$$x_i[t+1] = x_i[t] + \sum_{j \in \mathcal{R}_i^+[t]} a_{ij}[t] \left(x_j[t] - x_i[t] \right) \tag{4}$$

end for

Subsequently, we will relax the graph condition through the introduction of trusted nodes.

III. Resilient Containment Under Time-Invariant Networks

In this section, we present our methods to address the resilient containment problem under time-invariant digraphs. The following lemma shows a necessary condition for achieving resilient containment under time-invariant networks, which is a relatively stringent requirement for graph robustness.

Lemma 1: Consider a time-invariant digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A necessary condition for cooperative followers to achieve resilient containment is that each cooperative follower possesses at least $2f + 1$ in-neighbors, all of which are cooperative leaders. Furthermore, a cooperative follower satisfying this necessary condition will receive information from at least one leader at all time steps.

Proof: We start the proof with the first statement. Assume $|\mathcal{L}| \leq 2f$, and the state values of all other cooperative and noncooperative agents are p. Moreover, suppose that the state values of all f leaders are p_1, and the remaining $|\mathcal{L}| - f$ leader states are at p_2 with $p_1 < p < p_2$. Consider a cooperative follower $i \in \mathcal{F}_c$ possessing $|\mathcal{L}|$ leaders as in-neighbors. Through the leader-follower W-MSR algorithm, however, agent i will remove all the leaders out of its in-neighbor set $\mathcal{V}_i^+$, i.e., $|\mathcal{V}_i^+ \cap \mathcal{L}| = 0$. If noncooperative agents maintain their state values at p, the cooperative follower i will lack sufficient in-neighbors for update and its state value will retain at p. Thus, resilient containment cannot be achieved. For the second statement, we assume an agent $i \in \mathcal{F}_c$ possessing at least $2f + 1$ leaders as in-neighbors.

Through the leader-follower W-MSR algorithm, at most $2f$ in-neighbors are eliminated. Thus, at least one cooperative leader will retain in the in-neighbor set $\mathcal{V}_i^+$, and agent i will receive information from this leader at all time steps. ■

Remark 2: According to Lemma 1, to achieve resilient containment, each cooperative follower in the network should possess at least $2f+1$ leader in-neighbors. This condition is consistent with [27], in which the authors proved that resilient containment is achieved if the digraph of the MAS is strongly $(3f+1)$-robust w.r.t $\mathcal{L}$. Nevertheless, both $2f+1$ leader in-neighbors and strongly $(3f+1)$-robustness place high requirements on network connectivity and complexity, which are difficult to be satisfied, especially in large-scale distributed networks. Additionally, the paper [31] revealed that strong robustness induces a lower bound on the number of leaders, e.g., if the network is strongly r-robust w.r.t. $\mathcal{L}$, then $|\mathcal{L}| \geq r$.

Motivated by these observations, trusted nodes are introduced in this subsection to relax the high robustness requirements for graphs. Specifically, we set the leaders in the network as trusted nodes, i.e., $|\mathcal{L} \cap \mathcal{A}| = 0$, and the corresponding notion of *strongly trusted robustness* is defined as follows:

Definition 8 (Strongly trusted r-robust graph w.r.t. $\mathcal{S}_1$): Consider a time-invariant digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a nonempty subset $\mathcal{S}_1 \subseteq \mathcal{V}$. A network is said to be strongly trusted r-robust w.r.t. $\mathcal{S}_1$ if $|\mathcal{S}_1 \cap \mathcal{A}| = 0$, and for any nonempty subset $\mathcal{S}_2 \subseteq \mathcal{V} \backslash \mathcal{S}_1$, $\mathcal{S}_2$ is r-reachable, where $r \in \mathbb{Z}_{>0}$.

The authors of [27] derived that the digraph contains a directed spanning tree if the network is strongly $(3f+1)$-robust and resilient containment is further guaranteed. In the following theorem, we will prove that the network robustness requirement for resilient containment can be reduced from strongly $(3f+1)$-robust to strongly $(2f+1)$-robust by setting leaders as trusted nodes, while the lower bound on the number of leaders is also loosened.

Theorem 1: Consider a MAS described by a time-invariant digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose that the network satisfies the f-local model. Resilient containment is achieved if the network is strongly trusted $(2f+1)$-robust w.r.t. $\mathcal{L}$, and the cooperative agents execute the leader-follower W-MSR algorithm for update. Moreover, all cooperative followers will converge to $\mathcal{I}$ despite the existence of noncooperative agents.

Proof: The condition that $\mathcal{G}$ is strongly trusted $(2f+1)$-robust w.r.t. $\mathcal{L}$ indicates that there exists a nonempty subset $\mathcal{X}_1 \subseteq \mathcal{F}_c \subset \mathcal{V} \backslash \mathcal{L}$ such that $\forall i_1 \in \mathcal{X}_1, |\mathcal{V}_{i_1}^+ \backslash \mathcal{F}_c| \geq 2f+1$. Since the network satisfies the f-local attack model and $|\mathcal{L} \cap \mathcal{A}| = 0$, one obtains $|\mathcal{V}_{i_1}^+ \cap \mathcal{L}| \geq 2f+1$. Through the leader-follower W-MSR algorithm, at least one cooperative leader (labeled as l_1) is retained in the set of in-neighbors for i_1. Hence, there exists a cooperative leader l_1 that has a directed path to cooperative follower i_1.

Subsequently, let $\mathcal{F}_c^1 = \mathcal{F}_c \backslash \mathcal{X}_1$. If $\mathcal{F}_c^1 \neq \varnothing$, we can define a nonempty subset $\mathcal{X}_2 \subset \mathcal{F}_c^1$. Repeating the analysis of $\mathcal{X}_1$, one has $\forall i_2 \in \mathcal{X}_2$, $|\mathcal{V}_{i_2}^+ \backslash \mathcal{F}_c^1| \geq 2f+1$. Furthermore, through the leader-follower W-MSR algorithm, at least one cooperative agent $l_2 \in \mathcal{L}_c \cup \mathcal{X}_1$ is retained in the set of in-neighbors for i_2. This fact indicates that there exists a directed path from at least one cooperative leader to agent i_2, since l_2 is either a cooperative leader or a cooperative follower that utilizes the value of a cooperative leader for update.

By induction, one can let $\mathcal{F}_c^m = \mathcal{F}_c^{m-1} \backslash \mathcal{X}_m \neq \varnothing$ and $\mathcal{X}_{m+1} \subset \mathcal{F}_c^m$. Repeating the analysis above yields $\forall i_{m+1} \in \mathcal{X}_{m+1}$, $|\mathcal{V}_{i_{m+1}}^+ \backslash \mathcal{F}_c^m| \geq 2f+1$. This fact implies that at least one cooperative agent $l_m \in \mathcal{L}_c \cup \mathcal{X}_1 \cup \mathcal{X}_2 \cdots \cup \mathcal{X}_m$ is retained in the set of in-neighbors for i_{m+1}. Therefore, there always exists at least one cooperative leader that has a directed path to cooperative follower i_{m+1}.

The aforementioned induction can continue until all cooperative followers in $\mathcal{G}$ are involved. This is attainable since $\mathcal{G}$ is strongly trusted $(2f+1)$-robust w.r.t. $\mathcal{L}$. Using this derivation yields $\exists n \leq |\mathcal{F}_c|$, $\bigcup_{i=1}^n \mathcal{X}_i = \mathcal{F}_c$. This fact implies that $\mathcal{G}$ contains a directed spanning tree among cooperative agents, since the values of cooperative leaders can be transmitted to all cooperative followers in $\mathcal{G}$. By invoking [27, Theorem 1], resilient containment is guaranteed. Furthermore, since the network requirement is strongly $(2f+1)$-robust, the constraint on the minimum number of leaders is relaxed from $3f+1$ to $2f+1$. ■

IV. Resilient Containment Under Time-Varying Networks

By setting leaders as trusted nodes, we have relaxed the graph complexity to some extent. In practice, it is difficult to guarantee that the network satisfies certain graph conditions at each time step. To eliminate the above constraint, the resilient containment problem under time-varying digraphs is further investigated, which merely requires the network to be jointly and strongly trusted r-robust. We start with the following definition.

Definition 9 (Jointly r-reachable set [38]): Consider a time-varying digraph $\mathcal{G}[t] = (\mathcal{V}, \mathcal{E}[t])$ and a nonempty subset $\mathcal{S} \subseteq \mathcal{V}$. $\mathcal{S}$ is said to be jointly r-reachable if there exists an infinite sequence of uniformly bounded and non-overlapping time intervals $[t_h, t_{h+1})$ such that in each time interval $[t_h, t_{h+1})$, there exist a time step $t_j \in [t_h, t_{h+1})$ and an agent $i_j \in \mathcal{S}$ such that $|\mathcal{V}_{i_j}^+[t_j] \backslash \mathcal{S}| \geq r$, where $r \in \mathbb{Z}_{>0}$, $t_h = k_h T,\ h = 1, 2, \cdots$, and $k_1 = 0$.

Motivated by the aforementioned definition, we further present the notion of jointly and strongly trusted r-robust digraph as follows:

Definition 10 (Jointly and strongly trusted r-robust graph w.r.t. $\mathcal{S}_1$): Consider a time-varying digraph $\mathcal{G}[t] = (\mathcal{V}, \mathcal{E}[t])$ and a nonempty subset $\mathcal{S}_1 \subseteq \mathcal{V}$. $\mathcal{G}[t]$ is said to be jointly and strongly trusted r-robust w.r.t. $\mathcal{S}_1$ if $|\mathcal{S}_1 \cap \mathcal{A}| = 0$, and for any nonempty subset $\mathcal{S}_2 \subseteq \mathcal{V} \backslash \mathcal{S}_1$, $\mathcal{S}_2$ is jointly r-reachable, where $r \in \mathbb{Z}_{>0}$.

In the following part, we will prove that a MAS with the jointly and strongly trusted $(2f+1)$-robust network can achieve resilient containment under the f-local attack model. The following quantities are defined to derive this result.

$$M[t] = \max_{i \in \mathcal{L}_c, j \in \mathcal{F}_c} (x_i[t], x_j[t]),$$
$$m[t] = \min_{i \in \mathcal{L}_c, j \in \mathcal{F}_c} (x_i[t], x_j[t]). \tag{5}$$

The following lemma describes the monotonicity of $M[t]$ and $m[t]$.

Lemma 2: Consider a MAS described by a time-varying digraph $\mathcal{G}[t]=(\mathcal{V}[t],\mathcal{E})$. Suppose that the network is jointly and strongly trusted $(2f+1)$-robust w.r.t. $\mathcal{L}$ and satisfies the f-local attack model. If all cooperative agents execute the leader-follower W-MSR algorithm for update, then the following statements hold $\forall t\geq t_1$, where $t_1\geq 0$ is a given initial time step.

1) $M[t]$ is nonincreasing and $m[t]$ is nondecreasing.
2) $x_i[t]\in[m[t_1],M[t_1]]\ \forall i\in\mathcal{C}$.

Proof: For statement 1), we merely focus on cooperative followers since all the leaders are stationary and trusted. Consider a cooperative follower $j\in\mathcal{F}_c$. According to the definition in (5), we have $x_j[t]\in[m[t],M[t]]$. Since $\mathcal{G}[t]$ is strongly trusted $(2f+1)$-robust w.r.t. $\mathcal{L}$, and the network satisfies the f-local attack model, we know that agent j possesses at least $2f+1$ leader in-neighbors, all of which are trusted and not influenced by other f noncooperative in-neighbors. If the state values of these noncooperative agents exceed $[m[t],M[t]]$, then all of them will be removed through the leader-follower W-MSR algorithm. Thus, every remaining agent $l\in\mathcal{R}_j[t]$ satisfies $x_l[t]\in[m[t],M[t]]$ after the removal process, and the state value of agent j at time step $t+1$ is upper bounded by

$$\begin{aligned}x_j[t+1]&=a_{jj}[t]x_j[t]+\sum_{l\in\mathcal{R}_j[t]}a_{jl}[t]x_l[t]\\&\leq a_{jj}[t]M[t]+\sum_{l\in\mathcal{R}_j[t]}a_{jl}[t]M[t]\\&\leq M[t].\end{aligned}\tag{6}$$

Similarly, the state value $x_j[t+1]$ is lower bounded by

$$\begin{aligned}x_j[t+1]&=a_{jj}[t]x_j[t]+\sum_{l\in\mathcal{R}_j[t]}a_{jl}[t]x_l[t]\\&\geq a_{jj}[t]m[t]+\sum_{l\in\mathcal{R}_j[t]}a_{jl}[t]m[t]\\&\geq m[t].\end{aligned}\tag{7}$$

Note that j can be any cooperative follower in the network. Thus, we have $M[t+1]\leq M[t]$ and $m[t+1]\geq m[t]$ for all $t\geq t_1$. This fact implies that $M[t]$ is nonincreasing and $m[t]$ is nondecreasing.

For statement 2), we firstly consider a cooperative leader $i\in\mathcal{L}_c$. Synthesizing the assumption of stationary leaders and the definition of $M[t]$ yields $x_i[t]\leq M[t]$. Similarly, we can derive $x_i[t]\geq m[t]$. According to statement 1), we obtain $x_i[t]\in[m[t_1],M[t_1]]\ \forall i\in\mathcal{L}_c\ \forall t\geq t_1$. Now consider a cooperative follower $j\in\mathcal{F}_c$. According to the definition in (5), we have $x_j[t]\in[m[t],M[t]]$. Repeating the deductions in (6) and (7), we can obtain $x_j[t]\in[m[t_1],M[t_1]]$. ■

Remark 3: According to Lemma 2, we know that statements 1) and 2) are ensured by a combined action of three preset conditions: i) the jointly and strongly trusted robust graph, ii) the f-local model, and iii) the leader-follower MSR algorithm. To ensure these preset conditions, one should construct a proper network topology, limit the number of noncooperative agents in the in-neighbor set of each cooperative agent, and deploy the resilient algorithm on each agent.

In addition to proving the monotonicity of $M[t]$ and $m[t]$, Lemma 2 reveals that for a given initial time step t_1, the state values of cooperative agents at any subsequent time step $t\geq t_1$ are always within the interval constructed by the maximum and minimum state values of cooperative agents at t_1. In the following theorem, we will further demonstrate that the state values of cooperative followers converge to $\mathcal{I}=[m_L,M_L]$, i.e., the MAS achieves resilient containment control.

Theorem 2: Consider a MAS described by a time-varying digraph $\mathcal{G}[t]=(\mathcal{V}[t],\mathcal{E})$. Suppose that the network satisfies the f-local attack model. Resilient containment is achieved if the network is jointly and strongly trusted $(2f+1)$-robust w.r.t. $\mathcal{L}$, and the cooperative agents apply the leader-follower W-MSR algorithm for updating. Moreover, all cooperative followers will converge to $\mathcal{I}$ despite the existence of noncooperative agents.

Proof: According to statement 1) in Lemma 2, one obtains that $M[t]$ and $m[t]$ are bounded $\forall t\geq t_1$. Let A_M and A_m be the limits of $M[t]$ and $m[t]$, respectively. Then, the MAS achieves resilient containment if $A_M\leq M_L$ and $A_m\geq m_L$ hold simultaneously. Accordingly, the following four cases are discussed.

Case 1: $A_M>M_L$ and $A_m<m_L$.

Define $\lambda_0>0,\mu_0>0$ such that

$$\begin{aligned}A_M-\lambda_0&>M_L+\lambda_0\\A_m+\mu_0&<m_L-\mu_0.\end{aligned}\tag{8}$$

Due to $m_L<M_L$, it follows that

$$A_M-\lambda_0>A_m+\mu_0.\tag{9}$$

Moreover, let T_1 be the largest length of time intervals $[t_h,t_{h+1})$, and choose

$$\lambda=\frac{\omega^{|\mathcal{C}|T_1+1}}{1-\omega^{|\mathcal{C}|T_1+1}}\lambda_0,\ \mu=\frac{\omega^{|\mathcal{C}|T_1+1}}{1-\omega^{|\mathcal{C}|T_1+1}}\mu_0,\tag{10}$$

which satisfy $\lambda_0>\lambda>0$ and $\mu_0>\mu>0$. There exists a time step t_p such that

$$M[t]<A_M+\lambda,\quad m[t]>A_m-\mu,\quad\forall t\geq t_p.\tag{11}$$

due to the definition of convergence. It can be verified that

$$\lambda_0>\frac{\omega^{t_{|\mathcal{C}|+p}-t_p}}{1-\omega^{t_{|\mathcal{C}|+p}-t_p}}\lambda_0>\lambda>0\tag{12}$$

$$\mu_0>\frac{\omega^{t_{|\mathcal{C}|+p}-t_p}}{1-\omega^{t_{|\mathcal{C}|+p}-t_p}}\mu_0>\mu>0.\tag{13}$$

The strictly and monotonically decreasing sequences $\{\lambda_q\}$ and $\{\mu_q\}$ are further constructed as

$$\lambda_{q+1}=\omega\lambda_q-(1-\omega)\,\lambda\tag{14}$$

$$\mu_{q+1}=\omega\mu_q-(1-\omega)\,\mu,\tag{15}$$

where $q=0,\ldots,t_{|\mathcal{C}|+p}-t_p-1$. Substituting $q=t_{|\mathcal{C}|+p}-t_p-1$ into (14), we obtain

$$\begin{aligned}\lambda_{t_{|\mathcal{C}|+p}-t_p}&=\omega\lambda_{t_{|\mathcal{C}|+p}-t_p-1}-(1-\omega)\,\lambda\\&=\omega^2\lambda_{t_{|\mathcal{C}|+p}-t_p-2}-\omega\,(1-\omega)\,\lambda-(1-\omega)\,\lambda\end{aligned}$$

$$\begin{aligned}
&\vdots\\
&= \omega^{t_{|\mathcal{C}|+p}-t_p}\lambda_0 - \sum_{q=0}^{t_{|\mathcal{C}|+p}-t_p-1} \omega^q (1-\omega)\lambda\\
&= \omega^{t_{|\mathcal{C}|+p}-t_p}\lambda_0 - \left(1-\omega^{t_{|\mathcal{C}|+p}-t_p}\right)\lambda\\
&> 0. \qquad (16)
\end{aligned}$$

Similarly, we can derive

$$\mu_{t_{|\mathcal{C}|+p}-t_p} = \omega^{t_{|\mathcal{C}|+p}-t_p}\mu_0 - \left(1-\omega^{t_{|\mathcal{C}|+p}-t_p}\right)\mu > 0. \qquad (17)$$

These facts imply that $\lambda_q > 0$, $\mu_q > 0$ and $A_m + \mu_q \le A_M - \lambda_q$.

Accordingly, define two disjoint sequences $\mathcal{A}_1(t_p+q,\lambda_q)$ and $\mathcal{A}_2(t_p+q,\mu_q)$ as the node sets which include all cooperative and noncooperative agents that have values strictly larger than $A_M - \lambda_q$ and strictly smaller than $A_m + \mu_q$, respectively:

$$\begin{aligned}
\mathcal{A}_1(t_p+q,\lambda_q) &= \{i \in \mathcal{V} \mid x_i[t_p+q] > A_M - \lambda_q\}\\
\mathcal{A}_2(t_p+q,\mu_q) &= \{i \in \mathcal{V} \mid x_i[t_p+q] < A_m + \mu_q\}. \qquad (18)
\end{aligned}$$

Note that $\mathcal{A}_1(t_p+q,\lambda_q)$ and $\mathcal{A}_2(t_p+q,\mu_q)$ are disjoint according to the definition of λ_0 and μ_0. Their union and complementary sets are further defined as

$$\begin{aligned}
\mathcal{U}_{\mathcal{A}}(t_p+q,\lambda_q,\mu_q) &= \mathcal{A}_1(t_p+q,\lambda_q) \cup \mathcal{A}_2(t_p+q,\mu_q)\\
\mathcal{A}_3(t_p+q,\lambda_q,\mu_q) &= \mathcal{V}\backslash\mathcal{U}_{\mathcal{A}}(t_p+q,\lambda_q,\mu_q).
\end{aligned}$$

We will eventually demonstrate that

$$\begin{aligned}
&\left|\mathcal{U}_{\mathcal{A}}\left(t_{h+1},\lambda_{t_{h+1}-t_p},\mu_{t_{h+1}-t_p}\right)\cap\mathcal{C}\right|\\
&< \left|\mathcal{U}_{\mathcal{A}}\left(t_h,\lambda_{t_h-t_p},\mu_{t_h-t_p}\right)\cap\mathcal{C}\right|. \qquad (19)
\end{aligned}$$

To see this, we firstly reveal that

$$\{\mathcal{U}_{\mathcal{A}}(t_p+q+1,\lambda_{q+1},\mu_{q+1})\} \subset \{\mathcal{U}_{\mathcal{A}}(t_p+q,\lambda_q,\mu_q)\}$$

holds for all $q \ge 0$. Concentrated on different situations for cooperative followers, the following three subcases are discussed.

Subcase (i): Consider the cooperative follower $i \in \mathcal{A}_1(t_p+q,\lambda_q)$. Since $\mathcal{G}[t]$ is jointly and strongly trusted $(2f+1)$-robust w.r.t. $\mathcal{L}$, we know that agent i possesses at least $2f+1$ in-neighbors from outside $\mathcal{A}_1(t_p+q,\lambda_q)$, all of which are cooperative leaders. Thus, through the leader-follower W-MSR algorithm, at least one cooperative leader will be retained in the set of in-neighbors for agent i. According to Lemma 1, agent i will utilize at least one leader's state value for update at all time steps. Synthesizing the aforementioned analysis with the statement 2) in Lemma 2 yields that

$$\begin{aligned}
x_i[t_p+q+1] &= a_{ii}[t_p+q]x_i[t_p+q]\\
&\quad + \sum_{j\in\mathcal{R}_i[t_p+q]} a_{ij}[t_p+q]x_j[t_p+q]\\
&\le (1-\omega)M[t_p+q] + \omega M_L\\
&\le (1-\omega)(A_M+\lambda) + \omega(A_M-\lambda_q)\\
&= A_M - \omega\lambda_q + (1-\omega)\lambda\\
&= A_M - \lambda_{q+1}. \qquad (20)
\end{aligned}$$

Similarly, the state value of agent i at time step t_p+q+1 is lower bounded by

$$\begin{aligned}
x_i[t_p+q+1] &= a_{ii}[t_p+q]x_i[t_p+q]\\
&\quad + \sum_{j\in\mathcal{R}_i[t_p+q]} a_{ij}[t_p+q]x_j[t_p+q]\\
&\ge (1-\omega)m[t_p+q] + \omega m_L\\
&\ge (1-\omega)(A_m-\mu) + \omega(A_m+\mu_q)\\
&= A_m + \omega\mu_q - (1-\omega)\mu\\
&= A_m + \mu_{q+1}. \qquad (21)
\end{aligned}$$

According to the definition in (18), one has

$$\begin{aligned}
\mathcal{A}_1(t_p+q,\lambda_q) \cap \mathcal{A}_1(t_p+q+1,\lambda_{q+1}) &= \varnothing\\
\mathcal{A}_1(t_p+q,\lambda_q) \cap \mathcal{A}_2(t_p+q+1,\mu_{q+1}) &= \varnothing.
\end{aligned}$$

Therefore, it can be derived that

$$\mathcal{A}_1(t_p+q,\lambda_q) \subset \mathcal{A}_3(t_p+q+1,\lambda_{q+1},\mu_{q+1}).$$

Subcase (ii): Consider the cooperative follower $i \in \mathcal{A}_2(t_p+q,\mu_q)$. Similar to the analysis in Case 1, agent i possesses at least $2f+1$ leader in-neighbors and will utilize at least one leader's state value for update. Thus, one obtains

$$\mathcal{A}_2(t_p+q,\mu_q) \subset \mathcal{A}_3(t_p+q+1,\lambda_{q+1},\mu_{q+1}).$$

Subcase (iii): Consider the cooperative follower $i \in \mathcal{A}_3(t_p+q,\lambda_q,\mu_q)$. According to the definition in (18), one has $A_m + \mu_q \le x_i[t_p+q] \le A_M - \lambda_q$. Invoking the left inequality yields that

$$\begin{aligned}
x_i[t_p+q+1] &= a_{ii}[t_p+q]x_i[t_p+q]\\
&\quad + \sum_{j\in\mathcal{R}_i[t_p+q]} a_{ij}[t_p+q]x_j[t_p+q]\\
&\ge \omega(A_m+\mu_q) + (1-\omega)m[t_p+q]\\
&\ge \omega(A_m+\mu_q) + (1-\omega)(A_m-\mu)\\
&= A_m + \omega\mu_q - (1-\omega)\mu\\
&= A_m + \mu_{q+1}. \qquad (22)
\end{aligned}$$

Also, the state value $x_i[t_p+q+1]$ is upper bounded by

$$\begin{aligned}
x_i[t_p+q+1] &= a_{ii}[t_p+q]x_i[t_p+q]\\
&\quad + \sum_{j\in\mathcal{R}_i[t_p+q]} a_{ij}[t_p+q]x_{ij}[t_p+q]\\
&\le \omega(A_M-\lambda_q) + (1-\omega)M[t_p+q]\\
&\le \omega(A_M-\lambda_q) + (1-\omega)(A_M+\lambda)\\
&= A_M - \omega\lambda_q + (1-\omega)\lambda\\
&= A_M - \lambda_{q+1}. \qquad (23)
\end{aligned}$$

Therefore, one has

$$\mathcal{A}_3(t_p+q,\lambda_q,\mu_q) \subset \mathcal{A}_3(t_p+q+1,\lambda_{q+1},\mu_{q+1}).$$

Synthesizing the aforementioned three cases, one obtains

$$\{\mathcal{A}_1(t_p+q+1,\lambda_{q+1})\cap\mathcal{C}\}\subset\{\mathcal{A}_1(t_p+q,\lambda_q)\cap\mathcal{C}\}$$
$$\{\mathcal{A}_2(t_p+q+1,\mu_{q+1})\cap\mathcal{C}\}\subset\{\mathcal{A}_2(t_p+q,\mu_q)\cap\mathcal{C}\}.$$

Consequently, one derives

$$\{\mathcal{U}_\mathcal{A}(t_p+q+1,\lambda_{q+1},\mu_{q+1})\cap\mathcal{C}\}$$
$$\subset\{\mathcal{U}_\mathcal{A}(t_p+q,\lambda_q,\mu_q)\cap\mathcal{C}\}.$$

It follows that

$$|\mathcal{U}_\mathcal{A}(t_p+q+1,\lambda_{q+1},\mu_{q+1})\cap\mathcal{C}|$$
$$\le|\mathcal{U}_\mathcal{A}(t_p+q,\lambda_q,\mu_q)\cap\mathcal{C}|.$$

Applying the time interval $[t_h,t_{h+1})$ into the aforementioned analysis, we obtain

$$\left|\mathcal{U}_\mathcal{A}\left(t_{h+1},\lambda_{t_{h+1}-t_p},\mu_{t_{h+1}-t_p}\right)\cap\mathcal{C}\right|$$
$$\le\left|\mathcal{U}_\mathcal{A}\left(t_h,\lambda_{t_h-t_p},\mu_{t_h-t_p}\right)\cap\mathcal{C}\right|. \tag{24}$$

It remains to prove that the equality in (24) does not hold. We firstly consider the nonempty subset $\mathcal{A}_1(t_h,\lambda_{t_h-t_p})$. Since $\mathcal{G}[t]$ is jointly and strongly trusted $(2f+1)$-robust w.r.t. $\mathcal{L}$, there exists a cooperative agent $i_j\in\mathcal{A}_1(t_h,\lambda_{t_h-t_p})$ and a time step $t_j\in[t_h,t_{h+1})$ such that agent i_j possesses at least $2f+1$ leader in-neighbors at time step t_j. Firstly, we consider $i_j\in\mathcal{A}_1(t_h,\lambda_{t_h-t_p})$. If $i_j\notin\mathcal{A}_1(t_j,\lambda_{t_j-t_p})$, one obtains

$$\left|\mathcal{A}_1\left(t_{h+1},\lambda_{t_{h+1}-t_p}\right)\right|\le\left|\mathcal{A}_1(t_j,\lambda_{t_j-t_p})\right|<\left|\mathcal{A}_1(t_h,\lambda_{t_h-t_p})\right|$$

If $i_j\in\mathcal{A}_1(t_j,\lambda_{t_j-t_p})$, one derives $i_j\notin\mathcal{A}_1(t_{j+1},\lambda_{t_{j+1}-t_p})$ according to the analysis of *Subcase (i)* in Theorem 2. Therefore, it can be deduced that

$$\left|\mathcal{A}_1\left(t_{h+1},\lambda_{t_{h+1}-t_p}\right)\right|<\left|\mathcal{A}_1(t_j,\lambda_{t_j-t_p})\right|\le\left|\mathcal{A}_1(t_h,\lambda_{t_h-t_p})\right|$$

which yields that

$$\left|\mathcal{A}_1\left(t_{h+1},\lambda_{t_{h+1}-t_p}\right)\right|<\left|\mathcal{A}_1(t_h,\lambda_{t_h-t_p})\right|. \tag{25}$$

Similar argument can be derived if $i_j\in\mathcal{A}_2(t_h,\mu_{t_h-t_p})$. Thus, one has

$$\left|\mathcal{A}_2\left(t_{h+1},\mu_{t_{h+1}-t_p}\right)\right|<\left|\mathcal{A}_2(t_h,\mu_{t_h-t_p})\right|. \tag{26}$$

Synthesizing the results (25) and (26) yields that

$$\left|\mathcal{U}_\mathcal{A}\left(t_{h+1},\lambda_{t_{h+1}-t_p},\mu_{t_{h+1}-t_p}\right)\cap\mathcal{C}\right|$$
$$<\left|\mathcal{U}_\mathcal{A}\left(t_h,\lambda_{t_h-t_p},\mu_{t_h-t_p}\right)\cap\mathcal{C}\right|.$$

So far, we have proved (19), which results in the contradiction due to

$$\left|\mathcal{U}_\mathcal{A}\left(t_h,\lambda_{t_h-t_p},\mu_{t_h-t_p}\right)\cap\mathcal{C}\right|>0 \tag{27}$$

and

$$|\mathcal{U}_\mathcal{A}(t_p,\lambda_0,\mu_0)\cap\mathcal{C}|\le|\mathcal{C}|. \tag{28}$$

Specifically, the inequality (19) implies that in the infinite sequence $[t_h,t_{h+1})$, the strict decrease in the number of cooperative agents in the union set will continue indefinitely. However, the inequality (27) indicates that at time step h, there still exist some cooperative agents in the union set, while the inequality (28) means that the number of cooperative agents in the union set is finite. Consequently, the cooperative agents in the union set cannot decrease strictly and infinitely, and Case 1 is excluded from our discussion.

Case 2: $A_M>M_L$ and $A_m\ge m_L$.

Since $A_m\ge m_L$ has satisfied the condition of resilient containment, we only need to focus on $A_M>M_L$. Define $\varepsilon_0>0$ such that

$$A_M-\varepsilon_0>M_L+\varepsilon_0. \tag{29}$$

Also, we can construct the strictly and monotonically decreasing sequence $\{\varepsilon_q\}$ as

$$\varepsilon_{q+1}=\omega\varepsilon_q-(1-\omega)\varepsilon, \tag{30}$$

where

$$\varepsilon=\frac{\omega^{|\mathcal{C}|T_1+1}}{1-\omega^{|\mathcal{C}|T_1+1}}\varepsilon_0. \tag{31}$$

Then, we can define the following two disjoint sequences:

$$\mathcal{B}_1(t_l+q,\varepsilon_q)=\{i\in\mathcal{V}\mid x_i[t_l+q]>A_M-\varepsilon_q\}$$
$$\mathcal{B}_2(t_l+q,\varepsilon_q)=\{i\in\mathcal{V}\mid x_i[t_l+q]<M_L+\varepsilon_q\}, \tag{32}$$

where t_l is a time step such that

$$M[t]<A_M+\varepsilon,\quad m[t]>M_L-\varepsilon,\quad\forall t\ge t_l. \tag{33}$$

The union of the two sets defined in (32) is further defined as

$$\mathcal{U}_\mathcal{B}(t_l+q,\varepsilon_q)=\mathcal{B}_1(t_l+q,\varepsilon_q)\cup\mathcal{B}_2(t_l+q,\varepsilon_q).$$

By adopting a similar analysis process as Case 1, we will eventually derive

$$\left|\mathcal{U}_\mathcal{B}\left(t_{h+1},\varepsilon_{t_{h+1}-t_p}\right)\cap\mathcal{C}\right|<\left|\mathcal{U}_\mathcal{B}\left(t_h,\varepsilon_{t_h-t_p}\right)\cap\mathcal{C}\right|,$$

which also results in the contradiction. Case 2 is therefore refuted by the contradiction and cannot hold.

Case 3: $A_M\le M_L$ and $A_m<m_L$.

We only consider $A_m<m_L$ in this case. The analysis of resilient containment is similar to Case 2 and thus is omitted. We will eventually derive the contradiction and find that Case 3 is impossible.

Case 4: $A_M\le M_L$ and $A_m\ge m_L$.

In this case, no contradiction happens and resilient containment is guaranteed.

In summary, Case 4 is the only possible case under the conditions of Theorem 2. Therefore, resilient containment is achieved. ■

Remark 4: As stated in Theorems 1 and 2, the network should satisfy certain graph conditions to defend against malicious attacks. Specifically, the network should be strongly trusted $(2f+1)$-robust w.r.t $\mathcal{L}$ or jointly construct a strongly trusted $(2f+1)$-robust digraph to guarantee resilient containment under the f-local attack model. Compared to [27], we achieve resilient containment with relaxed graph robustness and fewer leader agents at the cost of setting trusted nodes. This result also implies a trade-off between setting trusted nodes and constructing more robust (yet more stringent) network structures. A detailed analysis of the trade-off will be one of our potential research directions in future work.

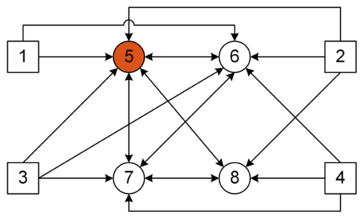


Fig. 1. A strongly 4-robust digraph with eight agents.

V. Case Studies

In this section, three numerical examples are provided to validate the theoretical results. The first case presents the achievement of resilient containment under time-invariant networks. With the introduction of trusted nodes, we can clearly see that the graph complexity is significantly reduced, while the lower bound on the number of leaders is also relaxed. For the second case, resilient containment is further achieved under time-varying networks, which requires that the digraph is jointly and strongly robust. Finally, we validate the scalability of the proposed method by extending the results to a larger-scale network.

A. Resilient Containment Under Time-Invariant Networks

Consider a MAS consisting of eight agents and described by Fig. 1, where $\mathcal{L} = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{5, 6, 7, 8\}$. It can be verified that the digraph is strongly 4-robust w.r.t. $\mathcal{L}$. According to Theorem 1, the digraph is able to handle the 1-local attack model. Thus, we let Agent 5 be noncooperative, whose motion is expressed as $x_5[t] = 10 \cdot \sin(t/5) + 2$. The initial state values of all eight agents are chosen as $[x_1[0], \ldots, x_8[0]]^{\mathrm{T}} = [2, 3, 2.5, 1.2, -2, 6, 2, 3.2]^{\mathrm{T}}$. Thus, we obtain $\mathcal{I} = [1.2, 3]$.

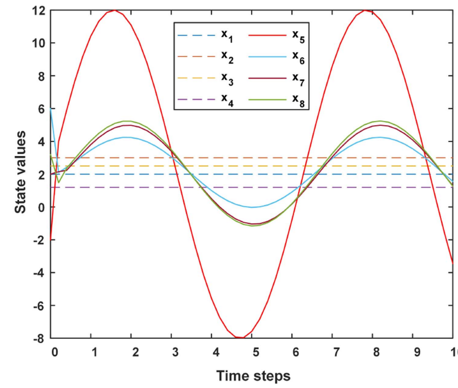
To start with, we apply the control methods proposed in [10] and [27] to the MAS separately. The simulation results are shown in Fig. 2(a)a and (b), respectively.

Fig. 2(a) displays the state value of each agent using the algorithm of [10]. It can be seen that cooperative followers are incapable of achieving resilient containment. Instead, they follow the noncooperative agent in sinusoidal motion, and the trajectory exceeds $\mathcal{I}$. This fact indicates the necessity of secure protocols and resilient algorithms.

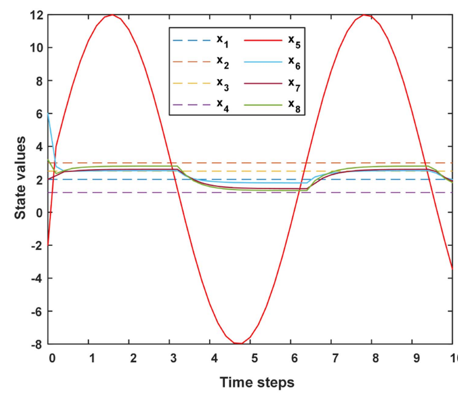
The state value of each agent with the resilient containment algorithm of [27] is illustrated in Fig. 2(b). It can be observed that resilient containment is guaranteed, since four cooperative followers can move within the safety interval constructed by the other four cooperative leaders. Nevertheless, the high graph robustness requirement makes the algorithm difficult to be conducted in practical scenarios. Specifically, in a strongly 4-robust digraph, each cooperative follower should possess at least four in-neighbors (agent 6 even possesses six in-neighbors), while the total directed edges reach $|\mathcal{E}| = 21$. Additionally, a strongly 4-robust digraph requires that half of the agents in Fig. 1 must be the leaders to drive the remaining agents into the safety interval, which is another constraint.

As a comparison, we next show the performance of the proposed notion of strongly trusted robustness. A strongly trusted 3-robust digraph is constructed in Fig. 3, where $\mathcal{L} = \mathcal{T} = \{2, 3, 4\}$ and $\mathcal{F} = \{1, 5, 6, 7, 8\}$. Note that Agent 1 changes from a leader to a follower due to the relaxation of the digraph. Let Agent 5 be noncooperative, whose motion is expressed as $x_5[t] = 10 \cdot \sin(t/5) + 2$. The network still satisfies the 1-local attack model. Other followers execute the leader-follower MSR algorithm. In the first scenario (with a fixed safety interval), we set the initial state values of eight agents to $[x_1[0], \ldots, x_8[0]]^{\mathrm{T}} = [2, 3, 2.5, 1.2, -2, 6, 2, 3.2]^{\mathrm{T}}$ and keep the state values of leaders unchanged. In the second scenario (with a varying safety interval), we set the state values of leaders as

$$[x_2[t], x_3[t], x_4[t]]^{\mathrm{T}} = \begin{cases} [3, 2.5, 1.2]^{\mathrm{T}}, & \text{if } t \in [0, 20), \\ [8, 7.5, 6.2]^{\mathrm{T}}, & \text{if } t \in [20, 40), \\ [-2, -3.5, -3.8]^{\mathrm{T}}, & \text{if } t \in [40, 50], \end{cases} \tag{34}$$



(a) Cooperative followers fail to converge to the fixed safety interval.



(b) Cooperative followers converge to the fixed safety interval.

Fig. 2. State values of agents in the strongly 4-robust digraph with (a) the algorithm of [10]; (b) the algorithm of [27]. Note that $\mathcal{I} = [1.2, 3]$.

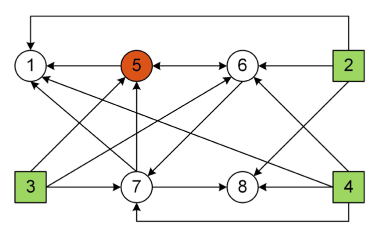
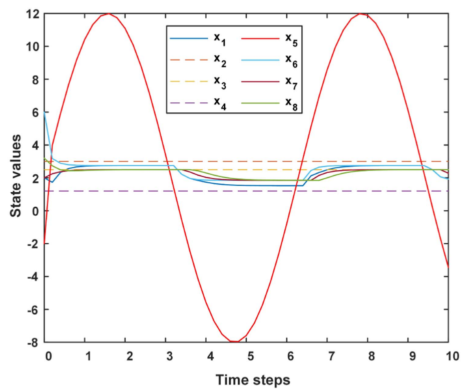
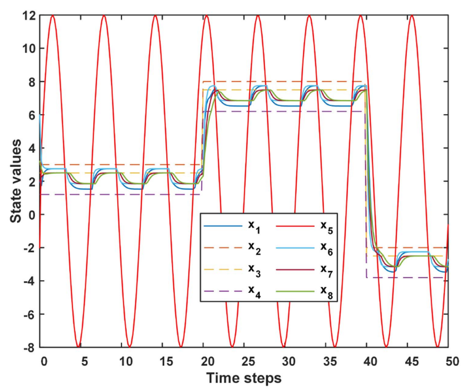


Fig. 3. A strongly trusted 3-robust digraph with eight agents.



(a) Cooperative followers converge to the fixed safety interval.



(b) Cooperative followers converge to the varying safety interval.

Fig. 4. State values of agents in the strongly trusted 3-robust digraph (a) the safety interval $\mathcal{I} = [1.2, 3]$; (b) the safety interval $\mathcal{I}_1 = [1.2, 3]$, $\mathcal{I}_2 = [6.2, 8]$, $\mathcal{I}_3 = [-3.8, -2]$.

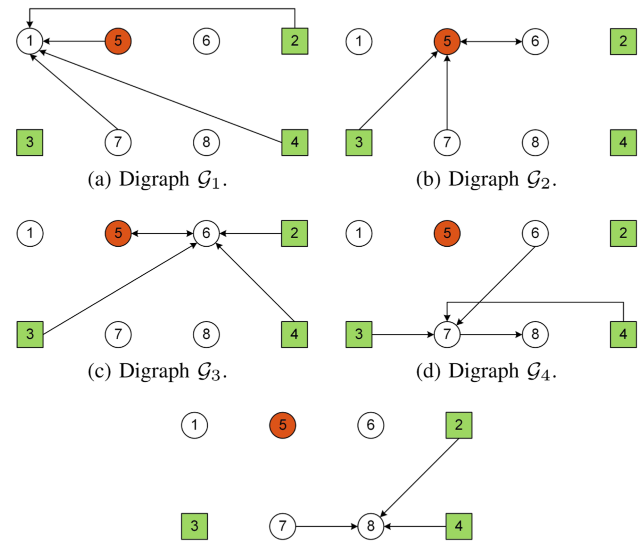
while the initial state values of followers are the same as that in the previous scenario.

The results are shown in Fig. 4(a) and (b), respectively. We observe that resilient containment is guaranteed in both cases, with safety intervals being adjusted arbitrarily. Under the strongly trusted 3-robust network, each cooperative follower has fewer in-neighbors, thus the number of total directed edges reduces to $|\mathcal{E}| = 17$. Furthermore, the constraint on the minimum number of leaders is relaxed, and we can use fewer leaders to drive more followers with a relaxed network complexity. The simulation result also validates Theorem 1.

B. Resilient Containment Under Time-Varying Networks

In this part, resilient containment is achieved under time-varying networks to reduce the graph robustness requirement at each time step. A jointly and strongly trusted 3-robust digraph with eight agents is shown in Fig. 5 and switches as follows:

$$\mathcal{G}[t] = \begin{cases} \mathcal{G}_1, & t = 5m, \\ \mathcal{G}_2, & t = 5m+1, \\ \mathcal{G}_3, & t = 5m+2, \quad m \in \mathbb{N} \\ \mathcal{G}_4, & t = 5m+3, \\ \mathcal{G}_5, & t = 5m+4, \end{cases} \tag{35}$$



(a) Digraph $\mathcal{G}_1$. (b) Digraph $\mathcal{G}_2$. (c) Digraph $\mathcal{G}_3$. (d) Digraph $\mathcal{G}_4$. (e) Digraph $\mathcal{G}_5$.

Fig. 5. A jointly and strongly trusted 3-robust digraph with eight agents.

Note that $\mathcal{L} = \mathcal{T} = \{2, 3, 4\}$ and $\mathcal{F} = \{1, 5, 6, 7, 8\}$. The noncooperative Agent 5 adheres to $x_5[t] = 10 \cdot \sin(t/5) + 2$. Other followers execute the leader-follower MSR algorithm. All five digraphs in Fig. 5 satisfy the 1-local attack model, but none of them is strongly trusted 3-robust w.r.t. $\mathcal{L}$. In the first scenario (with a fixed safety interval), we set the initial state values of eight agents to $[x_1[0], \ldots, x_8[0]]^{\mathrm{T}} = [2, 3, 2.5, 1.2, -2, 6, 2, 3.2]^{\mathrm{T}}$ and keep the state values of leaders unchanged. In the second scenario (with a varying safety interval), we set the state values of leaders as

$$[x_2[t], x_3[t], x_4[t]]^{\mathrm{T}} = \begin{cases} [-2, -3.5, -3.8]^{\mathrm{T}}, & \text{if } t \in [0, 20), \\ [8, 7.5, 6.2]^{\mathrm{T}}, & \text{if } t \in [20, 40), \\ [3, 2.5, 1.2]^{\mathrm{T}}, & \text{if } t \in [40, 50], \end{cases} \tag{36}$$

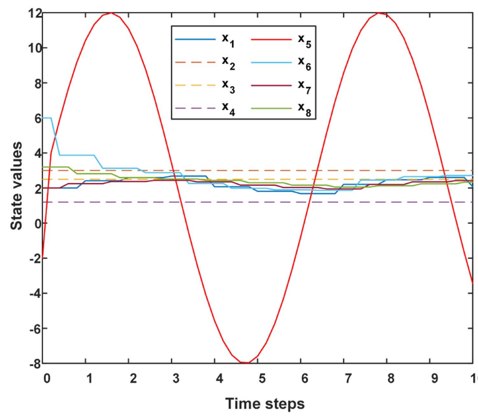
while the initial state values of followers are the same as that in the previous scenario.

The results are shown in Fig. 6(a) and (b), respectively, from which we observe that resilient containment is still guaranteed in both cases, while the safety intervals can be arbitrarily set by cooperative leaders.

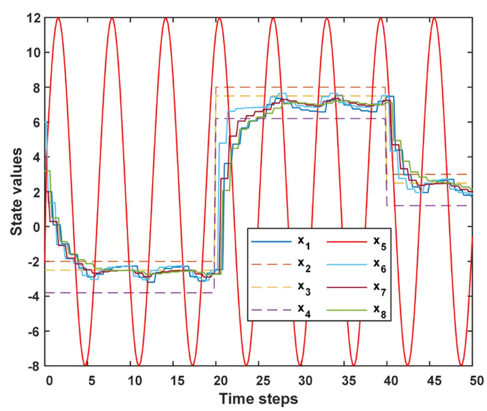
In addition to achieving resilient containment under time-varying networks, we calculate the total directed edges of each digraph and obtain $|\mathcal{E}_1| = 4, |\mathcal{E}_2| = 3, |\mathcal{E}_3| = 5, |\mathcal{E}_4| = 4, |\mathcal{E}_5| = 3$. The result reveals that the graph complexity at each time step is significantly reduced. Theorem 2 is also validated through this numerical result.

C. Scalability of the Proposed Method

In order to verify the scalability of the proposed method, we extend the results in Section V-B to a larger-scale MAS in this subsection.

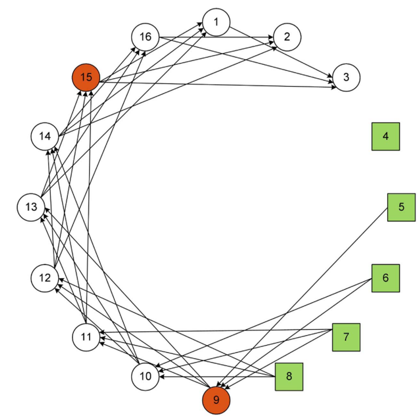


(a) Cooperative followers converge to the fixed safety interval.

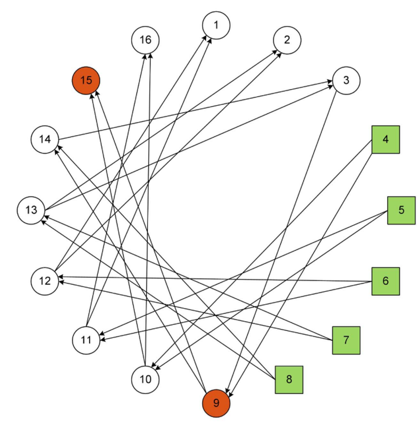


(b) Cooperative followers converge to the varying safety interval.

Fig. 6. State values of agents in the jointly and strongly trusted 3-robust digraph (a) the safety interval $\mathcal{I} = [1.2, 3]$; (b) the safety interval $\mathcal{I}_1 = [-3.8, -2], \mathcal{I}_2 = [6.2, 8], \mathcal{I}_3 = [1.2, 3]$.



(a) Digraph $\mathcal{G}_6$.



(b) Digraph $\mathcal{G}_7$.

Fig. 7. A jointly and strongly trusted 3-robust digraph with sixteen agents.

Consider a MAS consisting of sixteen agents and described by Fig. 7, where $\mathcal{L} = \mathcal{T} = \{4, 5, \ldots, 8\}$ and the remaining agents are followers. Assume that Fig. 7(a) and (b) switch between each other at each time step. It can be verified that the digraph is jointly and strongly 3-robust w.r.t. $\mathcal{L}$. Furthermore, let Agents 9 and 15 be noncooperative agents, whose trajectories are described as $x_9[t] = 10 \cdot \sin(t/5) + 2$ and $x_{15}[t] = 0.5 \cdot t$, respectively. Although there are two noncooperative agents in the network, the MAS still satisfies the 1-local attack model because at most one noncooperative agent exists in the in-neighbor set of each cooperative agent. The initial states of all cooperative followers are random values on the interval $[-10, 10]$, whereas the initial state values of trusted leaders are chosen as $[x_4[0], \ldots, x_8[0]]^{\mathrm{T}} = [0, 0, 0, 2, 2]^{\mathrm{T}}$.

The simulation result is shown in Fig. 8, from which we observe that the state values of cooperative followers (solid line) converge to the safety interval (green area), despite the influence of two noncooperative agents (dash dot line) in the network. Resilient containment is thereby guaranteed. The result also validates that the proposed method is scalable to large-scale networks.

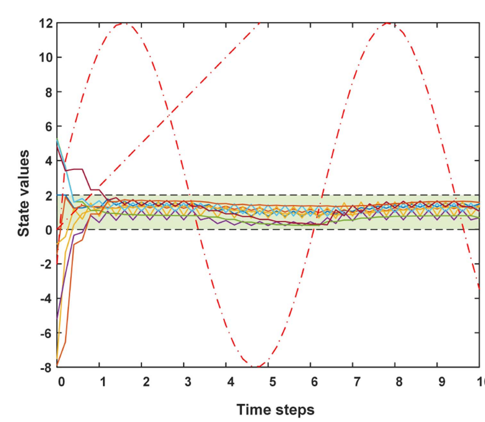


Fig. 8. State values of agents in a larger-scale MAS. The trajectories of cooperative followers (solid line) converge to the safety interval (green area), despite two noncooperative agents (dash dot line) exist in the network.

VI. Conclusion

In this paper, we consider resilient containment control for first-order MASs under time-invariant and time-varying networks. Despite the existence of non-cooperative agents in the network, secure protocols are designed to drive the cooperative followers into the safety interval constructed by the cooperative leaders. By setting leaders as trusted nodes, the notion of *strongly trusted robustness* is presented and resilient containment is achieved under time-invariant networks with relaxed graph conditions. The notion of *strongly and jointly trusted robustness* is further defined to guarantee resilient containment of the MAS under time-varying digraphs, where the network connection between agents is significantly reduced at each time step. Numerical examples are presented to illustrate the validity, superiority, and scalability of the obtained results.

Future work will apply the proposed resilient controller into practical systems and develop more advanced resilient algorithms to achieve resilient containment with relaxed graph robustness.

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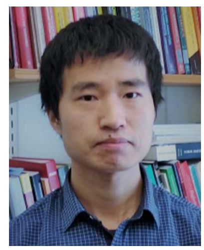

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